Exam I Review Math 202

Name: _____ Section: _____

Solving Systems of Equations

1. Solve this system, and write its solution set *both* in parametric form *and* in parametric vector form.

$$\begin{cases} x_1 + x_3 + 2x_4 = 3 \\ x_2 + x_3 + 3x_4 = 2 \\ 2x_1 - x_2 + x_3 + x_4 = 4 \end{cases}$$

Name: _____ Section: _____

2. Fill in the blanks, to complete the statement of Theorem 2:

   Theorem 2: The reduced echelon form of a linear system has three possible cases

   (a) The system has _____ solutions if _____

   (b) The system has _____ solutions if _____

   (c) The system has _____ solutions if _____

3. Give an example of the augmented matrix of a system of 3 equations in 3 variables ...
   (*Hint:* you can make the systems as simple as you like.)

   (a) with a unique solution

   (b) with no solutions

   (c) with infinitely many solutions.

Name: _____ Section: _____

4. Give an example of the augmented matrix of a system of 2 equations in 3 variables ... (*Hint:* you can make the systems as simple as you like.)

   (a) with no solutions

   (b) with infinitely many solutions.

5. Use Theorem 2 to prove that you cannot write a system of 2 equations in 3 variables that has a unique solution. (*Hint:* your argument must consider all relevant reduced-echelon form matrices.)

6. Where possible, give an example of the augmented matrix of a system of 3 equations in 3 variables with 0 solutions, 1 solution, and infinitely many solutions.

   If one of these cases are impossible, use Theorem 2 to explain why.

Name: _____ Section: _____

7. (No Computation) For each of the following, determine if it is in echelon form, reduced echelon form, or neither. If it is in echelon form, indicate which *columns* contain pivots.

(a)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

(d)

$$\begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Name: _____ Section: _____

Four Ways to Express the Same System

1. (a) Write the following augmented matrix as a vector equation, a matrix equation, and a system of linear equations.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -3 \\ 2 & 3 & 1 & 1 \end{array}\right]$$

(b) Is the system of equations consistent?

(c) Solve the above system. (Find the solution set).

(d) If the system is consistent, write down a particular solution.

Verify that this is a solution *two different ways* by plugging it into both *i.* the vector equation, and *ii.* the system of equation.

Name: _____ Section: _____

2. (a) Write the following augmented matrix as a vector equation, a matrix equation, and a system of linear equations.

$$\left[\begin{array}{cc|c} 0 & 4 & 6 \\ 5 & 2 & 10 \\ 3 & 3 & 6 \end{array}\right]$$

(b) Is the system of equations consistent?

(c) Solve the above system. (Find the solution set).

Verify that this is a solution *two different ways* by plugging it into both *i.* the vector equation, and *ii.* the system of equation.

Name: _____ Section: _____

3. (a) Write the following augmented matrix as a vector equation, a matrix equation, and a system of linear equations.

$$\left[\begin{array}{ccc|c} 3 & -3 & 3 & 3 \\ -1 & 2 & 1 & 3 \\ 2 & 8 & 2 & 2 \end{array}\right]$$

(b) Is the system of equations consistent?

(c) Solve the above system. (Find the solution set).

(d) If the system is consistent, write down a particular solution.

Verify that this is a solution *two different ways* by plugging it into both *i.* the vector equation, and *ii.* the system of equation.

Name: _____ Section: _____

Vectors and Vector Equations

1. (No Computation) Write down the formal definition of $\text{Span}\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$.

2. (No Computation) What is the graphical meaning of $\text{Span}\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$?

3. Describe the Span of the following sets of vectors:
   (a) $\text{Span}\{\vec{\mathbf{v}}\}$ where $\vec{\mathbf{v}} = \vec{\mathbf{0}}$

   (b) $\text{Span}\{\vec{\mathbf{v}}\}$ where $\vec{\mathbf{v}} \neq \vec{\mathbf{0}}$

   (c) $\text{Span}\{\vec{\mathbf{v}}, \vec{\mathbf{w}}\}$ where $\vec{\mathbf{v}}, \vec{\mathbf{w}} \neq \vec{\mathbf{0}}$ and $\vec{\mathbf{w}} = k\vec{\mathbf{v}}$ for some $k \in \mathbb{R}$.

   (d) $\text{Span}\{\vec{\mathbf{v}}, \vec{\mathbf{w}}\}$ where $\vec{\mathbf{v}}, \vec{\mathbf{w}} \neq \vec{\mathbf{0}}$ and $\vec{\mathbf{w}} \neq k\vec{\mathbf{v}}$ for every $k \in \mathbb{R}$.

Name: _____ Section: _____

4. Is the vector $\vec{\mathbf{b}} = \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$ in the span of the vectors $\vec{\mathbf{a}}_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$, $\vec{\mathbf{a}}_2 = \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}$, $\vec{\mathbf{a}}_3 = \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$?

5. Find h so that the vector the vector $\vec{\mathbf{b}} = \begin{bmatrix} 2 \\ h \\ 3 \end{bmatrix}$ in the span of the vectors

$\vec{\mathbf{a}}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $\vec{\mathbf{a}}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\vec{\mathbf{a}}_3 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$.

Name: _____ Section: _____

6. Write the Solution Set of the equation

$$x_1 \begin{bmatrix} 2 \\ -2 \\ -4 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ -1 \\ -6 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -4 \end{bmatrix}$$

in parametric vector form.

Name: _____ Section: _____

7. Do the vectors $\vec{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$, $\vec{\mathbf{v}}_= \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}$, and $\vec{\mathbf{v}}_3 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ span $\mathbb{R}^3$?

8. Find all possible h so that the vectors $\vec{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{\mathbf{v}}_1 = \begin{bmatrix} 3 \\ h \end{bmatrix}$ span $\mathbb{R}^2$.

Name: _____ Section: _____

9. (No Computation) Find 4 vectors that do span $\mathbb{R}^3$

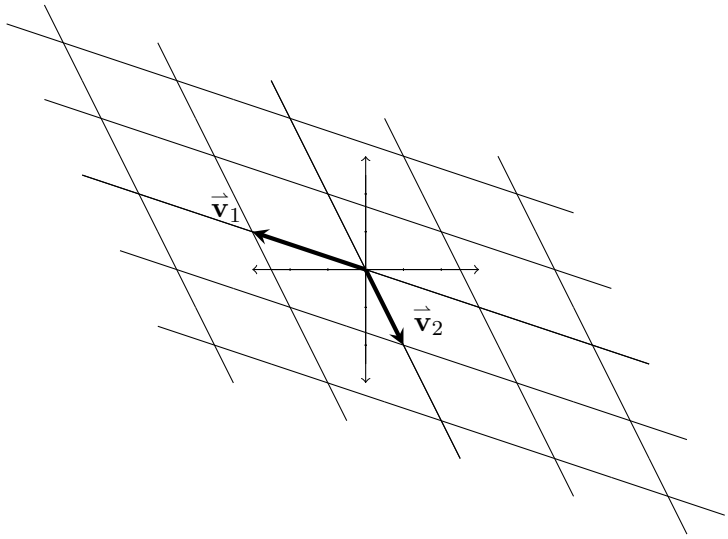
10. (No Computation) Find 4 vectors that do NOT span $\mathbb{R}^3$

11. (No Computation) Can you find 3 vectors that DO span $\mathbb{R}^4$? Why or why not?

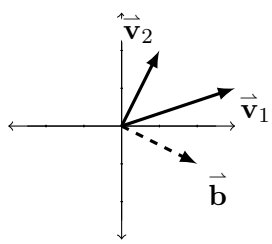
Name: _____ Section: _____

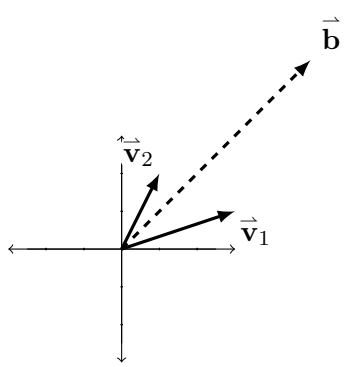
Visualizing Spans

1. Two vectors, $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$ are graphed below, along with the lattice of their linear combinations. Graph and label the vectors $\vec{\mathbf{a}} = 2\vec{\mathbf{v}}_1 + 1\vec{\mathbf{v}}_2$. $\vec{\mathbf{b}} = \vec{\mathbf{v}}_1 - \vec{\mathbf{v}}_2$, and $\vec{\mathbf{c}} = -1.5\vec{\mathbf{v}}_1 + 0.5\vec{\mathbf{v}}_2$,



2. Use the following vectors to find $s, t \in \mathbb{R}$ so that $\vec{\mathbf{b}} = s\vec{\mathbf{v}}_1 + t\vec{\mathbf{v}}_2$. In other words, write $\vec{\mathbf{b}}$ as a linear combination of $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$. It may be helpful to sketch lattice graph paper.

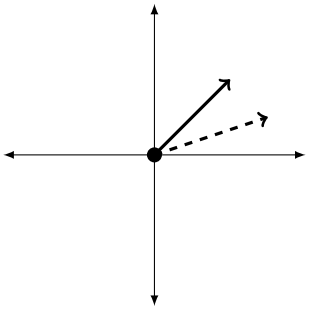




Name: _____ Section: _____

3. For each of the following vectors of $\mathbb{R}^2$, sketch the lattice graph paper of their integer combinations. Then, answer the questions given.

(a)

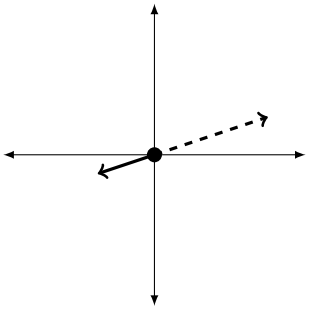


Define Span$\{\vec{\mathbf{b}}_1, \vec{\mathbf{b}}_2\}$

Is the span is a point, line, or a plane?

Do the vectors span $\mathbb{R}^2$?

(b)

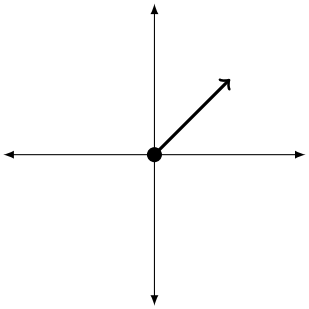


Define Span$\{\vec{\mathbf{b}}_1, \vec{\mathbf{b}}_2\}$

Is the span is a point, line, or a plane?

Do the vectors span $\mathbb{R}^2$?

(c)

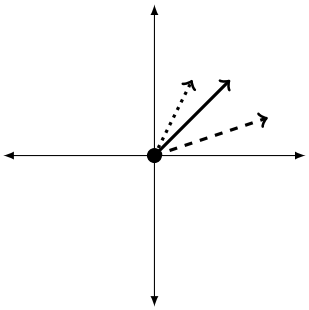


Define Span$\{\vec{\mathbf{b}}_1\}$

Is the span is a point, line, or a plane?

Do the vectors span $\mathbb{R}^2$?

(d)



Define Span$\{\vec{\mathbf{b}}_1, \vec{\mathbf{b}}_2, \vec{\mathbf{b}}_3\}$

Is the span is a point, line, or a plane?

Do the vectors span $\mathbb{R}^2$?

Name: _____ Section: _____

Matrices and Matrix Equations

1. If $A = [\vec{\mathbf{a}}_1, \ldots, \vec{\mathbf{a}}_n]$ and $\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, write down the formal definition of the matrix product

$$A\vec{\mathbf{x}} = [\vec{\mathbf{a}}_1, \ldots, \vec{\mathbf{a}}_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} =$$

2. Compute the following matrix products. If the product is undefined, explain why.

   (a) $\begin{bmatrix} 1 & 2 \\ -2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

3. Suppose that $A = [\vec{\mathbf{a}}_1 \ldots \vec{\mathbf{a}}_n]$ is an $m \times n$ matrix. Rephrase the sentence "The columns of A span $\mathbb{R}^m$" as a statement about vectors.

4. Suppose that A is an 5×7 matrix, and that $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$.

   Find j and k so that $\vec{\mathbf{x}} \in \mathbb{R}^j$ and $\vec{\mathbf{b}} \in \mathbb{R}^k$.

Name: _____ Section: _____

5. Let $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$. Does the system $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has a solution for every $\vec{\mathbf{b}} \in \mathbb{R}^2$?

   You must justify your answer.

6. Let $A = \begin{bmatrix} 3 & 6 & 3 & 0 \\ -3 & -9 & 0 & 3 \\ 0 & 3 & -3 & -3 \end{bmatrix}$.

   Do the columns of $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ span $\in \mathbb{R}^3$. You must justify your answer.

Name: _____ Section: _____

7. (No Computation) Fill in the blanks to state Theorem 4 in terms of pivots.

   Theorem 4: The columns of an $m \times n$ matrix A span $\mathbb{R}^m$

   if and only if there is a pivot _____

8. (No Computation) Write down a 3×3 matrix A whose columns span $\mathbb{R}^3$.

9. (No Computation) Write down a 3×3 matrix A whose columns do not span $\mathbb{R}^3$.

10. (No Computation) Can you write down a 2×3 matrix A whose columns span $\mathbb{R}^2$? Justify your answer.

11. (No Computation) Can you write down a 3×2 matrix A whose columns span $\mathbb{R}^3$? Justify your answer.

Name: _____ Section: _____

Counting the Number of Possible Solutions

1. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

   (a) How many solutions are there to the equation $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$?

   (b) Let $\vec{\mathbf{b}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. How many solutions are there to the equation $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$?

   (c) Let $\vec{\mathbf{c}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. How many solutions are there to the equation $A\vec{\mathbf{x}} = \vec{\mathbf{c}}$?

   (d) How many solutions are *possible* to $A\vec{\mathbf{x}} = \vec{\mathbf{d}}$, when considering *all* vectors $\vec{\mathbf{d}} \in \mathbb{R}^3$? You must justify your answer.

Name: _____ Section: _____

2. Let $A = \begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix}$

   (a) How many solutions are there to the equation $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$?

   (b) Let $\vec{\mathbf{b}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. How many solutions are there to the equation $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$?

   (c) Let $\vec{\mathbf{c}} = \begin{bmatrix} 5 \\ -15 \end{bmatrix}$. How many solutions are there to the equation $A\vec{\mathbf{x}} = \vec{\mathbf{c}}$?

   (d) How many solutions are *possible* to $A\vec{\mathbf{x}} = \vec{\mathbf{d}}$, when considering *all* vectors $\vec{\mathbf{d}} \in \mathbb{R}^2$? You must justify your answer.

Name: _____ Section: _____

3. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

(a) How many solutions are there to the equation $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$?

(b) Let $\vec{\mathbf{b}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. How many solutions are there to the equation $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$?

(c) Let $\vec{\mathbf{c}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. How many solutions are there to the equation $A\vec{\mathbf{x}} = \vec{\mathbf{c}}$?

(d) How many solutions are *possible* to $A\vec{\mathbf{x}} = \vec{\mathbf{d}}$, when considering *all* vectors $\vec{\mathbf{d}} \in \mathbb{R}^2$? You must justify your answer.

Name: _____ Section: _____

Linear Dependence and Independence

1. (No Computation) Write down the formal definition of Linear **Dependence** of vectors $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$.

2. (No Computation) Give an example of a non-trivial dependence relation between $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$. Use this dependence relation to explain the graphical meaning of "Linear **Dependence**".

3. (No Computation) What is the graphical meaning of the set $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$ being Linearly **Independent**?

Name: _____ Section: _____

4. Determine if the following set of vectors is linearly dependent or independent.

$\vec{\mathbf{a}}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\vec{\mathbf{a}}_2 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ $\vec{\mathbf{a}}_3 = \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix}$

If the set $\{\vec{\mathbf{a}}_1, \vec{\mathbf{a}}_2, \vec{\mathbf{a}}_3\}$ is *dependent*, give an example of a particular nontrivial dependence relation.

Name: _____ Section: _____

5. Find h so that the vectors $\vec{\mathbf{a}}_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$, $\vec{\mathbf{a}}_2 = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$, and $\vec{\mathbf{a}}_3 = \begin{bmatrix} 2 \\ h \\ 1 \end{bmatrix}$ are linearly **independent**.

Name: _____ Section: _____

6. (No Computation) Suppose that $x_1\vec{\mathbf{v}}_1 + x_2\vec{\mathbf{v}}_2 + x_3\vec{\mathbf{v}}_3 = \vec{\mathbf{0}}$ has a unique solution.
   Is $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$ Linearly **Independent**?

7. (No Computation) Suppose that $x_1\vec{\mathbf{v}}_1 + x_2\vec{\mathbf{v}}_2 + x_3\vec{\mathbf{v}}_3 = \vec{\mathbf{0}}$ has infinitely many solutions.
   Is $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$ Linearly **Independent**?

8. (No Computation) Does $x_1\vec{\mathbf{v}}_1 + x_2\vec{\mathbf{v}}_2 + x_3\vec{\mathbf{v}}_3 = \vec{\mathbf{0}}$ always have a solution? Why or why not?

9. **Shortcuts to Recognize Linear *Dependence***
   - *If* one column of A is a multiple of another, *then* the columns of A are linearly dependent.
   - *If* $\{\vec{\mathbf{a}}_1, \ldots, \vec{\mathbf{a}}_n\}$ contains $\vec{\mathbf{0}}$, *then* $\{\vec{\mathbf{a}}_1, \ldots, \vec{\mathbf{a}}_n\}$ is linearly dependent.
   - *If* an $m \times n$ matrix A has more columns than rows (if $n > m$),
     then the columns of A are linearly dependent.

10. **Shortcut to Recognize Linear *Independence***
    - A set of two vectors $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2\}$ (i.e. if $n = 2$) is independent
      if and only if neither vector is a multiple of the other.
    - A set of one vector $\{\vec{\mathbf{v}}_1\}$ (i.e. if $n = 1$) is independent *if and only if* $\vec{\mathbf{v}}_1 \neq \vec{\mathbf{0}}$

Name: _____ Section: _____

11. For each of the following sets of vectors determine if they are linearly independent. You must justify each answer.

(a) $\mathcal{U} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

(b) $\mathcal{U} = \left\{ \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 6 \end{bmatrix} \right\}$

(c) $\mathcal{U} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 6 \end{bmatrix} \right\}$

(d) $\mathcal{U} = \left\{ \begin{bmatrix} 2 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix} \right\}$

(e) $\mathcal{U} = \left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \right\}$